Extra Credit 11

Prove that the graph above cannot be drawn with a single self-intersection.

Let **v** be the number of vertices and **E** be the number of edges of Peterson. We have v = 10, E = 15. Inspecting the graph, we see that:

* Each edge is included in exactly two faces (the unbounded area outside of the whole graph is considered as a face).
* Each cycle has a length of 5 (edges) or greater, denoted c ≥ 5.

These two facts yield:

2e ≥ 5f. (1)

By Euler’s formula, we have

v – e + f = 2, (2)

where **f** is the number of faces, **e** is number of edges of the planar graph.

Substituting **f** from (2) into (1), we have:

2e ≥ 5 (2 – v + e)

2e ≥ 10 – 5v + 5e

-3e ≥ 10 – 5v

e ≤ – (10 – 5v) / 3 (3)

The original number of edges E = 15. Let k be the number of self-intersections such that,

e = E – k. (4)

Substituting (3) into (4), E = 15, v = 10. We have,

*k* ≥ E + (10 – 5v) / 3

*k* > 15 + (10 – 50/ 3)

*k* ≥ 5/3

Since k is an integer, *k* ≥ 2.

Hence, the Peterson Graph requires at least two self-intersections to be drawn on a plane.